Objectives

1. Derivation of Boltzmann's Equation.
2. Collision Terms.

Conside a phase space function $f_j(\mathbf{v_j}, \mathbf{r_j}, t)$ of a particle j. Infinitesimal change in $\mathbf{v_j}$ and $\mathbf{r_j}$ can be expressed in terms of dt that $d\mathbf{v_j} = \frac{\mathbf{F_j}}{m_j}dt$ and $d\mathbf{r_j} = \mathbf{v_j}dt$. By Taylor expansion,

$$df_j = \left[\left(\frac{\partial f_j}{\partial t}\right) + \mathbf{v_j} \cdot \nabla_{\mathbf{r}} f_j + \frac{\mathbf{F_j}}{m_j} \cdot \nabla_{\mathbf{v_j}} f_j\right]dt \tag{1}$$

Then, we have:

$$\left(\frac{\partial f_j}{\partial t}\right) + \mathbf{v_j} \cdot \nabla_{\mathbf{r}} f_j + \frac{\mathbf{F_j}}{m_j} \cdot \nabla_{\mathbf{v_j}} f_j = \left(\frac{df_j}{dt}\right) \tag{2}$$

$\frac{df_j}{dt}$ can be known by understanding the collision between gas particles because the gain and the loss of f_j with time is dependent on the scattering due to collisions. With the normalization to the number density of particle j:

$$\int_{-\infty}^{\infty} f_j d\mathbf{v_j} = \rho_j \tag{3}$$

$$\langle \mathbf{v_j} \rangle = \frac{1}{\rho_j} \int_{-\infty}^{\infty} f_j \mathbf{v_j} d\mathbf{v_j}$$

With $\mathbf{v_0}$ as the flow velocity:

$$\mathbf{v_0}(\mathbf{r}, t) = \frac{\sum_j \rho_j m_j \langle \mathbf{v_j} \rangle}{\sum_j \rho_j m_j}$$

The peculiar velocity is defined as $(\mathbf{v_j} - \mathbf{v_0})$.

One interesting observation is that:

$$\frac{\sum_j m_j \int_{-\infty}^{\infty} (\mathbf{v_j} - \mathbf{v_0}) f_j d\mathbf{v_j}}{\sum_j \rho_j m_j} = 0$$

With the proof that:

$$\frac{\sum_j m_j \int_{-\infty}^{\infty} \mathbf{v_j} f_j d\mathbf{v_j}}{\sum_j \rho_j m_j} - \frac{\mathbf{v_0} \sum_j m_j \int_{-\infty}^{\infty} f_j d\mathbf{v_j}}{\sum_j \rho_j m_j}$$
$$= \frac{\sum_j \rho_j m_j \langle \mathbf{v_j} \rangle}{\sum_j \rho_j m_j} - \mathbf{v_0} = 0$$

Now, for a particle j, we can write for f_j:

$$\frac{\partial f_j}{\partial t} + \mathbf{v_j} \cdot \nabla_{\mathbf{r}} f_j + \frac{\mathbf{F_j}}{m_j} \cdot \nabla_{\mathbf{v_j}} f_j = \text{collision contribution}$$

The task becomes figuring out this collision cotribution.

Understanding a binary collision

$$m_1 + m_2 = m_1' + m_2' \tag{4}$$

Without any reaction, then $m_1 = m_1'$ and $m_2 = m_2'$. Then, the momentum balance gives us:

$$m_1\mathbf{v_1} + m_2\mathbf{v_2} = m_1\mathbf{v_1'} + m_2\mathbf{v_2'} \tag{5}$$

$$0.5m_1v_1^2 + 0.5m_2v_2^2 = 0.5m_1'v_1'^2 + 0.5m_2'v_2'^2 \tag{6}$$

To make our lives easier, the velocities can be rewritten in terms of relative velocity ($\mathbf{v_r}$) and center-of-mass velocity ($\mathbf{v_c}$).

$$\mathbf{v_r} = \mathbf{v_1} - \mathbf{v_2}$$

$$\mathbf{v_c} = \frac{m_1\mathbf{v_1} + m_2\mathbf{v_2}}{m_1 + m_2}$$

Consider the momentum balance:

$$\begin{aligned} m_1\mathbf{v_1} + m_2\mathbf{v_2} &= m_1\mathbf{v_1'} + m_2\mathbf{v_2'} \\ &= m_1\left(\mathbf{v_c} + \frac{m_2}{m_1+m_2}\mathbf{v_r}\right) + m_2\left(\mathbf{v_c} - \frac{m_1}{m_1+m_2}\mathbf{v_r}\right) \\ &= (m_1+m_2)\mathbf{v_c} = (m_1+m_2)\mathbf{v_c'} \end{aligned}$$

This proves that $\mathbf{v_c} = \mathbf{v_c'}$. Consider the following definition of $\mathbf{v_1}$ and $\mathbf{v_2}$:

$$\mathbf{v_1} = \mathbf{v_c} + \frac{m_2}{m_1 + m_2}\mathbf{v_r}$$
$$\mathbf{v_2} = \mathbf{v_c} - \frac{m_1}{m_1 + m_2}\mathbf{v_r}$$
$$v_1^2 = v_c^2 + \frac{m_2}{m_1 + m_2}\mathbf{v_c} \cdot \mathbf{v_r} + \frac{m_2^2}{(m_1 + m_2)^2}v_r^2$$
$$v_2^2 = v_c^2 - \frac{m_1}{m_1 + m_2}\mathbf{v_c} \cdot \mathbf{v_r} + \frac{m_1^2}{(m_1 + m_2)^2}v_r^2$$

The energy balance equation then becomes:

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}(m_1 + m_2)v_c^2 + \frac{1}{2}\mu v_r^2$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$. This means that $v_r = v_r'$. In conclusion, during an elastic collision, the magnitude of v_c and v_r remain unchanged. Only the direction of $\mathbf{v_r}$ have changed, such that $\mathbf{v_r} \cdot \mathbf{v_r}' = v_r^2 \cos\chi$.

We know that $d\mathbf{v_1}d\mathbf{v_2} = |\det(\mathbf{J})|d\mathbf{v_r}d\mathbf{v_c}$, with $\mathbf{J}$:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{v_1}}{\partial \mathbf{v_c}} & \frac{\partial \mathbf{v_1}}{\partial \mathbf{v_r}} \\ \frac{\partial \mathbf{v_2}}{\partial \mathbf{v_c}} & \frac{\partial \mathbf{v_2}}{\partial \mathbf{v_r}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{m_2}{m_1+m_2} \\ 1 & -\frac{m_1}{m_1+m_2} \end{bmatrix}$$

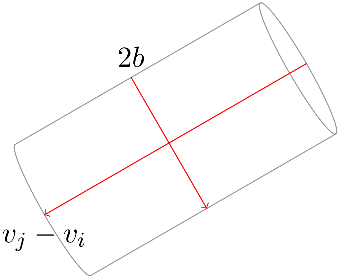
$|\det(\mathbf{J})| = 1$. Therefore, $d\mathbf{v_r}d\mathbf{v_c}$. Similarly, for the post-collisional velocities, $d\mathbf{v_1}'d\mathbf{v_2}' = d\mathbf{v_r}'d\mathbf{v_c}$. In spherical coordinate, we get:

$$d\mathbf{v_r}' = v_r'^2 \sin\theta dv_r' d\theta d\psi = d\mathbf{v_r} = v_r^2 \sin\theta dv_r d\theta d\psi$$

We come to the conclusion that $d\mathbf{v_1}d\mathbf{v_2} = d\mathbf{v_1}'d\mathbf{v_2}'$.

Collision Terms

We have to figure out the number of collisions with another particle i experienced by particle j per unit time. In one unit time, if we assume that all particles j are stationary, then particle i has traveled a distance of $|\mathbf{v_i} - \mathbf{v_j}|$. For one particle j,



Number of collisions per unit time

$$= \sum_i 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} f_i v_r b db d\mathbf{v_i}$$

Since $\rho_i = \int_{-\infty}^{\infty} f_i d\mathbf{v_i}$, and the volume is $2\pi \int_{-\infty}^{\infty} \int_0^{\infty} v_r b db d\mathbf{v_i}$.

The probable number of particle j in an element of $d\mathbf{r}$ within the range of $\mathbf{v_j}$ to $\mathbf{v_j} + d\mathbf{v_j}$ is $f_j d\mathbf{r} d\mathbf{v_j}$

Therefore, the probable number of collision for particle j per unit time in an element of $d\mathbf{r}$ within the range of $\mathbf{v_j}$ to $\mathbf{v_j} + d\mathbf{v_j}$.

$$f_j d\mathbf{r} d\mathbf{v_j} \sum_i 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} f_i v_r b db d\mathbf{v_i}$$

Similarly, consider the post-collisional velocity.

$$f_j' d\mathbf{r} d\mathbf{v_j}' \sum_i 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} f_i' v_r' b db d\mathbf{v_i}'$$

$$= f_j' d\mathbf{r} d\mathbf{v_j} \sum_i 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} f_i' v_r b db d\mathbf{v_i}$$

The term associated with the post-collisional velocity is responsible for the gain, whereas that of the pre-collisional velocity is responsible for the loss. Therefore, we have:

$$\text{Collision contribution} = \sum_i 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} (f_j' f_i' - f_j f_i) v_r b db d\mathbf{v_i}$$