Objectives

1. Basic introduction to vector and tensor.
2. Continuum mechanics.

$$\mathsf{P} = \begin{bmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}, \ \nabla_{\mathbf{r}} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix}$$

Alternatively, we can do the math in index notation:

$$\nabla_{\mathbf{r}} \cdot \mathbf{v} = \partial_k v_k = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\mathbf{vv} = v_k v_l = \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix}$$

Operation on pressure tensor:

$$\nabla_{\mathbf{r}} \cdot \mathsf{P} = \partial_k p_{kl} = \begin{bmatrix} \frac{\partial}{\partial x}(p_{xx} + p_{xy} + p_{xz}) \\ \frac{\partial}{\partial y}(p_{yx} + p_{yy} + p_{yz}) \\ \frac{\partial}{\partial z}(p_{zx} + p_{zy} + p_{zz}) \end{bmatrix}$$

k and l can be x, y, z as we have three directions.

Finally, we have the dydiac product.

$$\mathbf{v}\mathbf{v} : \nabla_{\mathbf{r}}\mathbf{v} = v_k v_l \partial_l v_k = \sum_{k=x,y,z} \sum_{l=x,y,z} v_k v_l \frac{\partial}{\partial l} v_k$$

And it is a scalar.

Three fundamental laws:

1. Mass conservation.
2. Momentum conservation.
3. Energy conservation

Mass Conservation

$$m = \int_V \rho(\mathbf{r}, t) d\mathbf{r}$$

$$\frac{dm}{dt} = \int_V \frac{\partial \rho(\mathbf{r}, t)}{\partial t} d\mathbf{r}$$

Applying divergence theorem:

$$\frac{dm}{dt} = -\int_V \nabla_{\mathbf{r}} \cdot (\rho \mathbf{v}) d\mathbf{r}$$

Then, we have:

$$\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{r}} \cdot (\rho \mathbf{v}) = 0$$

Momentum Conservation

$$m\mathbf{v} = \int_V \rho(\mathbf{r}, t)\mathbf{v} d\mathbf{r}$$

$$\frac{dm\mathbf{v}}{dt} = \int_V \frac{\partial \rho \mathbf{v}}{\partial t} d\mathbf{r} = -\int_V \nabla_{\mathbf{r}} \cdot (\rho \mathbf{v}\mathbf{v} + \mathsf{P}) d\mathbf{r}$$

Then we get:

$$\mathbf{v}\frac{\partial \rho}{\partial t} + \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla_{\mathbf{r}} \cdot (\rho \mathbf{v}) + \rho \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{v} + \nabla_{\mathbf{r}} \cdot \mathsf{P} = 0$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{v} + \nabla_{\mathbf{r}} \cdot \mathsf{P} = 0$$

Define the material derivative as $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}}$.

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla_{\mathbf{r}} \cdot \mathsf{P} = 0$$

Energy Conservation

$$E = \frac{mv^2}{2} + m\tilde{E} = \int_V \left(\frac{\rho v^2}{2} + \rho\tilde{E}\right) d\mathbf{r}$$

$$\frac{dE}{dt} = \int_V \left[\rho\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t} + \frac{v^2}{2}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\tilde{E}}{\partial t} + \tilde{E}\frac{\partial\rho}{\partial t}\right] d\mathbf{r}$$

$$= -\int_V \nabla_{\mathbf{r}}\cdot\left(\frac{\rho v^2}{2}\mathbf{v} + \mathbf{q} + \mathbf{v}\cdot\mathsf{P} + \rho\tilde{E}\mathbf{v}\right) d\mathbf{r}$$

Then we get:

$$\rho[\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t} + \frac{\partial\tilde{E}}{\partial t}] + [\frac{v^2}{2} + \tilde{E}]\frac{\partial\rho}{\partial t} + \nabla_{\mathbf{r}}\cdot\left(\frac{\rho v^2}{2}\mathbf{v} + \mathbf{q} + \mathbf{v}\cdot\mathsf{P} + \rho\tilde{E}\mathbf{v}\right) = 0$$

$$\left[\frac{v^2}{2} + \tilde{E}\right]\left[\frac{\partial\rho}{\partial t} + \nabla_{\mathbf{r}}\cdot(\rho\mathbf{v})\right] + \rho\mathbf{v}\cdot\frac{D\mathbf{v}}{Dt} + \mathbf{v}\nabla_{\mathbf{r}} : \mathsf{P}$$

$$+\nabla_{\mathbf{r}}\cdot\mathbf{q} + \mathsf{P} : \nabla_{\mathbf{r}}\mathbf{v} + \rho\frac{D\tilde{E}}{Dt} = 0$$

$$\nabla_{\mathbf{r}} \cdot \mathbf{q} + \mathsf{P} : \nabla_{\mathbf{r}} \mathbf{v} + \rho \frac{D\tilde{E}}{Dt} = 0$$

We can see the origin of the dydiac product term more clearly by writing $\nabla_{\mathbf{r}} \cdot \mathbf{v} \cdot \mathsf{P}$ in index notation.

$$\nabla_{\mathbf{r}} \cdot (\mathbf{v} \cdot \mathsf{P}) = \partial_l v_k p_{kl} = v_k \partial_l p_{lk} + p_{kl} \partial_l v_k = \mathbf{v} \nabla_{\mathbf{r}} : \mathsf{P} + \mathsf{P} : \nabla_{\mathbf{r}} \mathbf{v}$$

where we have assumed tensor is symmetrical that $p_{kl} = p_{lk}$.